Exercises for Friday, second hour

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The two envelopes paradox (Cover and Thomas, Exercise 6.11) I put b cents into one envelope and $2b$ cents into another, and then randomly give you one of the two envelopes. You now hold an envelope containing the random amount X, and there is another envolope on the table which contains the random amount Y.

1. Compute $\mathsf{E}[\,Y/X\,]$.
2. I give you the possbility of swapping envelope. By what factor is your capital expected to grow if you swap?
3. Explain this weird result.

A frequentist bias-variance tradeoff Suppose μ is a fixed but unknown constant somewhere on the real number line, and that a number x is drawn randomly from a uniform distribution on $[\mu,\, \mu + 1]$. We consider three different estimators of μ for this problem:

1. $\hat{\mu}_1(x) = 0$;
2. $\hat{\mu}_2(x) = x - {}^1\!/\!_2$;
3. $\hat{\mu}_3(x) = {}^x\!/\!_2 - {}^1\!/\!_4$.

Find the expected value of the squared error, $(\hat{\mu}(x) - \mu)^2$, for each of these estimators, averaged over all possible data points. For which values of μ is $\hat{\mu}_2$ better than $\hat{\mu}_3$?

A Bayesian bias-variance tradeoff A parameter θ is drawn from a uniform distribution on the unit interval; a bent coin with parameter θ is then flipped a single time, giving either the observation $k = 0$ or $k = 1$. We estimate θ using an estimator of the form

$$\hat{\theta}(k) = \frac{k + \alpha}{1 + 2\alpha}.$$

What is the mean squared error of this estimator for $\alpha = 0, 1, 2$, averaging both over the data points k and over the distribution of θ?